

Invariant measures of reflected stochastic delay differential equations with jumps *

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Abstract In this paper, we consider a class of multi-dimensional reflected stochastic delay differential equations with jumps. Based on the existence and uniqueness of the strong solution to the equation, we prove that the Markov semigroup generated by the segment process corresponding to the solution admits a unique invariant measure on the Skorohod space when the coefficients of equation satisfy a class of monotone conditions. Finally, we establish a relationship between the regulator and the local time of the solution and discuss a local time property at large time under the stationary setting .

Keywords: Stochastic delay differential equations; jump reflection; invariant measure; local time.
MSC: 60J60; 60K10.

1 Introduction and framework

We take a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; t \geq 0), \mathbb{P})$ carrying an $(\mathcal{F}_t; t \geq 0)$ -adapt $n \in \mathbb{N}$ -dimensional Brownian motion $W = (W_j(t); t \geq 0)_{n \times 1}$ and an $(\mathcal{F}_t; t \geq 0)$ -adapt Poisson random measure $(N((0, t] \times A); A \in \mathcal{B}(\mathcal{E}))$ with intensity $(\nu(A)t; A \in \mathcal{B}(\mathcal{E}))$, where $t > 0$ and $\mathcal{E} = \mathbb{R}^d / \{\mathbf{0}\}$. Here the filtration $(\mathcal{F}_t; t \geq 0)$ is assumed to satisfy the usual conditions. In this paper, we consider the following $d \in \mathbb{N}$ -dimensional reflected stochastic delay differential equation with jumps (RSDDEJ for short):

$$(1) \quad \begin{cases} dX(t) &= b(t, X(t), X(t-\tau))dt + \sigma(t, X(t), X(t-\tau))dW(t) \\ &+ \int_{\mathcal{E}} g(t, X(t-), X((t-\tau)-), \rho) \tilde{N}(d\rho, dt) + dK(t), & \text{on } t \geq 0, \\ X(t) &= \xi(t) \in \mathbb{R}_+^d, & \text{on } t \in [-\tau, 0], \end{cases}$$

where $\tau > 0$ is a deterministic delay level, the initial data $\xi(\cdot) \in D([-\tau]; \mathbb{R}_+^d)$, the space of all right-continuous functions with left limits from $[-\tau, 0]$ to \mathbb{R}_+^d and $\tilde{N}(d\rho, dt) := N(d\rho, dt) - \nu(d\rho)dt$ defines the compensated version of the Poisson measure N . Here the characteristic measure ν of N is a σ -finite measure on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$.

We write the solution process $X = (X(t); t \geq -\tau)$, the coefficient functions (b, g) in RSDDEJ (1) as $d \times 1$ column vectors and the diffusion coefficient σ as a $d \times n$ -matrix. Namely $X(t) = (X^i(t))_{d \times 1}$,

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$b(\cdot, \cdot, \cdot) = (b_i(\cdot, \cdot, \cdot))_{d \times 1}$, $\sigma(\cdot, \cdot, \cdot) = (\sigma_{ij}(\cdot, \cdot, \cdot))_{d \times n}$, $g(\cdot, \cdot, \cdot) = (g_i(\cdot, \cdot, \cdot))_{d \times 1}$. In addition, $K = (K^i(t); t \geq 0)_{d \times 1}$ is a d -dimensional nonnegative process, which is called the regulator for the d -dimensional solution process X at the orthant. Moreover, the regulator K can be uniquely determined by the following properties up to a positive constant factor (see Harrison [H86]):

- (a) For $i = 1, 2, \dots, d$, the paths of $t \rightarrow K^i(t)$ are non-decreasing, right-continuous with left limits (r.c.l.l. for short) and $K^i(0) = K^i(0-) = 0$;
- (b) For all $t \geq 0$, it holds that

$$(2) \quad \int_0^t \langle X(s), dK(s) \rangle = 0,$$

where $\langle x, y \rangle = \sum_{i=1}^d x_i y_i$ for $x = (x_i)_{d \times 1}$ and $y = (y_i)_{d \times 1} \in \mathbb{R}^d$.

For any $d \times n$ -matrix $a = (a_{ij})_{d \times n}$, define $\|a\| = \sqrt{\text{Tr}[aa^T]}$, where a^T is the transpose of a and $\text{Tr}[aa^T]$ denotes the trace of the matrix aa^T . Define $|x| = \sqrt{\langle x, x \rangle}$ for any $x = (x_i)_{d \times 1} \in \mathbb{R}^d$. We work in the following assumptions concerning coefficient functions (b, σ, g) in RSDDEJ (1) throughout the paper:

(A1) there exists a constant $\alpha > 0, \alpha_1 > \alpha_2 > 0$ such that

$$2\langle x, b(t, x, y) \rangle + \|\sigma(t, x, y)\|^2 + \int_{\mathcal{E}} |g(t, x, y, \rho)|^2 \nu(d\rho) \leq \alpha - \alpha_1 |x|^2 + \alpha_2 |y|^2,$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}_+^d$.

(A2) there exist two constants $\beta_1 > \beta_2 > 0$ such that

$$\begin{aligned} & 2\langle x - \hat{x}, b(t, x, y) - b(t, \hat{x}, \hat{y}) \rangle + \|\sigma(t, x, y) - \sigma(t, \hat{x}, \hat{y})\|^2 \\ & + \int_{\mathcal{E}} |g(t, x, y, \rho) - g(t, \hat{x}, \hat{y}, \rho)|^2 \nu(d\rho) \leq -\beta_1 |x - \hat{x}|^2 + \beta_2 |y - \hat{y}|^2, \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $x, \hat{x}, y, \hat{y} \in \mathbb{R}_+^d$.

An illustrative example for the condition **(A1)** and **(A2)** is to take the drift coefficient $b(t, x, y) = -\gamma(t)x + \theta(t)y$ with $\gamma(t), \theta(t) > 0$. We assume that $\gamma_* = \inf_{t \geq 0} \gamma(t)$ and $\theta^* = \sup_{t \geq 0} \theta(t)$ are finite. For $(t, x, \hat{x}, y, \hat{y}) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathbb{R}_+^d$, it holds that

$$2\langle x - \hat{x}, b(t, x, y) - b(t, \hat{x}, \hat{y}) \rangle \leq -(2\gamma_* - \varepsilon^2)|x - \hat{x}|^2 + \frac{|\theta^*|^2}{\varepsilon^2}|y - \hat{y}|^2,$$

for any $\varepsilon > 0$. Moreover the coefficient functions (σ, g) are assumed to satisfy $\sigma(t, \mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $g(t, \mathbf{0}, \mathbf{0}, \rho) = \mathbf{0}$ for all $(t, \rho) \in \mathbb{R}_+ \times \mathcal{E}$ and the following Lipschitzian-type conditions:

$$\begin{aligned} \|\sigma(t, x, y) - \sigma(t, \hat{x}, \hat{y}, \rho)\|^2 & \leq \ell_\sigma(t)(|x - \hat{x}|^2 + |y - \hat{y}|^2), \\ |g(t, x, y, \rho) - g(t, \hat{x}, \hat{y}, \rho)|^2 & \leq \ell_g(t, \rho)(|x - \hat{x}|^2 + |y - \hat{y}|^2), \end{aligned}$$

where $\ell_\sigma(t)$ and $\ell_g(t, \rho)$ are positive functions satisfying $\ell_{\sigma, g}^* := \sup_{t \geq 0} (\ell_\sigma(t) + \int_{\mathcal{E}} \ell_g(t, \rho) \nu(d\rho)) < +\infty$. If $\gamma_* - \ell_{\sigma, g}^* > \theta^*$, we can always find a constant $\varepsilon > 0$ such that $\alpha_1 := 2\gamma_* - \varepsilon^2 - \ell_{\sigma, g}^* > \alpha_2 := \frac{|\theta^*|^2}{\varepsilon^2} + \ell_{\sigma, g}^*$. Thus, the assumptions **(A1)** and **(A2)** are satisfied.

Given the Brownian motion W and Poisson random measure N , we call the $(\mathcal{F}_t; t \geq 0)$ -pair of r.c.l.l. processes $(X, K) = ((X(t); t \geq -\tau), (K(t); t \geq 0))$ is a strong solution to (1), if they solve the following stochastic integral equation:

$$(3) \quad \begin{cases} X(t) = \xi(0) + \int_0^t b(s, X(s), X(s-\tau))ds + \int_0^t \sigma(s, X(s), X(s-\tau))dW(s) \\ \quad + \int_0^t \int_{\mathcal{E}} g(s, X(s-), X((s-\tau)-), \rho) \tilde{N}(d\rho, ds) + K(t) \in \mathbb{R}_+^d, & \text{on } t \geq 0, \\ X(t) = \xi(t) \in \mathbb{R}_+^d, & \text{on } t \in [-\tau, 0], \end{cases}$$

and the \mathbb{R}_+^d -valued regulator K satisfies the properties **(a)** and **(b)**.

We first have the following remark concerning the existence and uniqueness of the strong solution to RSDDEJ (1).

Remark 1.1. *Under the assumptions **(A1)** and **(A2)**, RSDDEJ (1) admits a unique strong solution defined as above. As a matter of fact, the existence of the unique strong solution to (1) can be guaranteed by the following weaker conditions than **(A1)** and **(A2)**, namely*

(A1') *there exists a positive constant $\bar{\alpha}$ such that*

$$2\langle x, b(t, x, y) \rangle + \|\sigma(t, x, y)\|^2 + \int_{\mathcal{E}} |g(t, x, y, \rho)|^2 \nu(d\rho) \leq \bar{\alpha}(1 + |x|^2 + |y|^2),$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}_+^d$.

(A2') *there exist two positive constants $\bar{\beta}_1, \bar{\beta}_2$ such that*

$$2\langle x - \hat{x}, b(t, x, y) - b(t, \hat{x}, \hat{y}) \rangle + \|\sigma(t, x, y) - \sigma(t, \hat{x}, \hat{y})\|^2 \\ + \int_{\mathcal{E}} |g(t, x, y, \rho) - g(t, \hat{x}, \hat{y}, \rho)|^2 \nu(d\rho) \leq \bar{\beta}_1 |x - \hat{x}|^2 + \bar{\beta}_2 |y - \hat{y}|^2,$$

for all $t \in \mathbb{R}_+$ and $x, \hat{x}, y, \hat{y} \in \mathbb{R}_+^d$.

*The proof is similar to that of [K99] and [vRS], we omit it here. Our aim is to use the conditions **(A1)** and **(A2)** to study invariant measures for RSDDEJ (1).*

For $i = 1, 2, \dots, d$, let $K^{i,c}(t) = K^i(t) - \sum_{s \leq t} \Delta K^i(s)$ be the continuous part of the i th-regulator $K^i(t)$, where the t -time jump's size $\Delta K^i(t) = K^i(t) - K^i(t-)$ with left limit $K^i(t-) := \lim_{s \uparrow t} K^i(s)$. It will be seen that the continuous counterpart $K^{i,c} = (K^{i,c}; t \geq 0)$ behaves like the local time of the i th-element X^i of the solution process X when X^i is treated as a r.c.l.l. semimartingale (see Section 4 in the current paper). However, the jump of K^i happens when $X^i - K^i$ jumps down below barrier zero due to the appearance of some negative jump. This phenomenon is usually called “jump reflection” in the literature (e.g., Slomiński and Wojciechowski [SW10] and Nam [N10]). Moreover, the corresponding jump's size of the i th-regulator is given by

$$(4) \quad \Delta K^i(t) = \left[\int_{\{t\}} \int_{\mathcal{E}} g_i(s, X(s-), X((s-\tau)-), \rho) N(d\rho, ds) + X^i(t-) \right]^{-},$$

where $[x_i]^- = \max\{-x_i, 0\}$ for $x_i \in \mathbb{R}$ and $[x]^- = (x_i^-)_{d \times 1}$ for $x \in \mathbb{R}^d$. Write $K^c = (K^{i,c}(t); t \geq 0)_{d \times 1}$. From the “jump reflection”, we also have that, for all $t \geq 0$, it holds that

$$(5) \quad \int_0^t \langle X(s), dK^c(s) \rangle = 0.$$

The similar one-sided Lipschitzian condition and monotone condition as **(A1)** and **(A2)** has been discussed in Bao, et al. [BTY09] for stochastic delay equation without jump reflection, in Marin-Rubio and Real [MRR] and Zhang [Z94] for reflected stochastic differential equations without jumps. In particular, the Picard's successive approximation used in Xu and Zhang [XZ09] can deal with the existence and uniqueness of strong solutions to RSDDEJ (1) when the regulator K is described as a local time. However, due to the existing of negative jumps in (1), the regulator K has jumps whose sizes can be identified by (4). Let $(X_n; n = \{0\} \cup \mathbb{N})$ be the corresponding Picard's approximating sequence to the d -dimensional solution processes. Then, the successive approximation to the jump $\Delta K(t)$ can be established through (4), namely

$$\Delta K_n(t) = \left[\int_{\{t\}} \int_{\mathcal{E}} g(s, X_{n-1}(s-), X_{n-1}((s-\tau)-), \rho) N(d\rho, ds) + X_{n-1}(t-) \right]^{-}, \quad n \in \mathbb{N}.$$

See Proposition 2.4 in Slomiński and Wojciechowski [SW10] for more details. The literature concerning stochastic delay differential equations with (or without) jumps is extensive (see e.g., [A10], [FØ11], [KW10], [ØS10], [ØS07], [HMS11], [RRG06], [W11], and the references therein). Recently Kinnally and Williams [KW10] discussed the existence and uniqueness of stationary solutions to a class of reflected SDDE driven by Brownian motions. The stability property in distribution of Brownian-driven reflected Markov-modulated SDDE was considered in Bo and Yuan [BY11]. The stability in distribution implies the existence and uniqueness of invariant measures for the corresponding segment processes. To the best of our knowledge, it seems that there exists not much literature to investigate SDDE with jump reflection.

An outline of the paper is as follows. In Section 2, we establish an estimate for the second-order moment associated to the segment process of the solution to RSDDEJ (1) and consider an exponential integrability of the solution when the drift and diffusion coefficients are uniformly bounded. The existence and uniqueness of invariant measures associated with d -dimensional segment process is proved in Section 3. In Section 4, we discuss the relationship between the regulator K and the local time related to the strong solution process and a local time property in the stationary setting.

Additional Notation. For any $f(\cdot) \in D(I; \mathbb{R}_+^d)$, $\|f\|_I := \sup_{t \in I} |f(t)|$, where $I \subset \mathbb{R}$. For the d -dimensional solution process $X = (X(t); t \geq -\tau)$, the corresponding d -dimensional segment process $(X_t; t \geq 0)$ is defined as $X_t(\theta) = X(t + \theta)$ with $-\tau \leq \theta \leq 0$, correspondingly $\|X_t\| := \sup_{-\tau \leq \theta \leq 0} |X(t + \theta)|$. Throughout the paper, we use the conventions:

$$\int_c^d := \int_{(c,d]}, \quad \text{and} \quad \int_c^\infty := \int_{(c,\infty)},$$

for any real numbers $c < d$.

2 Moment estimates of the solution

This section concentrates on the estimates of the second-order moment related to the d -dimensional segment process and the exponential moment for the d -dimensional solution process $X = (X(t); t \geq -\tau)$ to RSDDEJ (1).

Before discussing these moment estimates, we first present the following auxiliary results which will serve to establish final moment estimates.

Lemma 2.1. *Let $X = (X(t); t \geq -\tau)$ be the strong solution to RSDDEJ (1). Then, for any $F(\cdot) \in C^2(\mathbb{R}_+^d)$, it holds that*

$$\begin{aligned}
F(X(t)) &= F(\xi(0)) + \int_0^t \langle \nabla F(X(s)), b(s, X(s), X(s-\tau)) \rangle ds \\
&\quad + \int_0^t \langle \nabla F(X(s)), \sigma(s, X(s), X(s-\tau)) dW(s) \rangle + \int_0^t \langle \nabla F(X(s)), dK^c(s) \rangle \\
&\quad + \int_0^t \int_{\mathcal{E}} \langle \nabla F(X(s)), g(s, X(s), X(s-\tau), \rho) \rangle \tilde{N}(d\rho, ds) \\
&\quad + \frac{1}{2} \int_0^t \text{Tr} [(\sigma \sigma^T)(s, X(s), X(s-\tau)) D^2 F(X(s))] ds \\
&\quad + \int_0^t \int_{\mathcal{E}} [F([X(s-) + g(s, X(s-), X((s-\tau)-), \rho)]^+) - F(X(s-)) \\
&\quad - \langle \nabla F(X(s-)), g(s, X(s-), X((s-\tau)-), \rho) \rangle] N(d\rho, ds),
\end{aligned} \tag{6}$$

where $[x]^+ = (x_i^+)_{d \times 1}$ and $x_i^+ = \max\{x_i, 0\}$ with $x = (x_i)_{d \times 1} \in \mathbb{R}^d$, $\nabla F(x)$ denotes the gradient of $F(x)$, $D^2 F(x)$ is the $d \times d$ -matrix of second-order partial derivatives of $F(x)$ and $K^c(t) = (K^{i,c}(t))_{d \times 1}$ corresponds to the continuous component of the regulator $K(t) = (K^i(t))_{d \times 1}$ with $t \geq 0$.

Proof. By virtue of Itô formula with jumps (see e.g., Protter [P04]), we have

$$\begin{aligned}
F(X(t)) &= F(\xi(0)) + \int_0^t \langle \nabla F(X(s-)), dX(s) \rangle \\
&\quad + \frac{1}{2} \int_0^t \text{Tr} [(\sigma \sigma^T)(s, X(s), X(s-\tau)) D^2 F(X(s))] ds \\
&\quad + \sum_{0 < s \leq t} [F(X(s-) + \Delta X(s)) - F(X(s-)) - \langle \nabla F(X(s-)), \Delta X(s) \rangle].
\end{aligned} \tag{7}$$

For $i = 1, 2, \dots, d$, define the process with pure jumps:

$$Y^i(t) = \int_0^t \int_{\mathcal{E}} g_i(s, X(s-), X((s-\tau)-), \rho) N(d\rho, ds), \quad t \geq 0. \tag{8}$$

In terms of (1), the random jump amplitude of the i th-element X^i ($i = 1, 2, \dots, d$) is clearly given by $\Delta X^i(t) = \Delta Y^i(t) + \Delta K^i(t)$ for $t > 0$. Using the following key representation of the jump's size of the i th-regulator K^i with $i = 1, 2, \dots, d$ (see (4)):

$$\Delta K^i(t) = [\Delta Y^i(t) + X^i(t-)]^- \quad t > 0, \tag{9}$$

we arrive at

$$\Delta X^i(t) = \Delta Y^i(t) + [\Delta Y^i(t) + X^i(t-)]^- =: \varphi(X^i(t-), \Delta Y^i(t)), \quad t > 0. \tag{10}$$

The function $\varphi(x, y) = (\varphi(x_i, y_i))_{d \times 1}$ with $x = (x_i)_{d \times 1} \in \mathbb{R}_+^d$ and $y = (y_i)_{d \times 1} \in \mathbb{R}^d$, where $\varphi(x_i, y_i) = -x_i \mathbb{1}_{\{x_i + y_i \leq 0\}} + y_i \mathbb{1}_{\{x_i + y_i > 0\}}$ for $i = 1, 2, \dots, d$. Using the equality $x_i + \varphi(x_i, y_i) = [x_i + y_i]^+$ and substitute the following equality into the equality (7),

$$F(X(s-) + \Delta X(s)) - F(X(s-)) - \langle \nabla F(X(s-)), \Delta X(s) \rangle$$

$$= F([X(s-) + \Delta Y(s)]^+) - F(X(s-)) - \langle \nabla F(X(s-)), \Delta Y(s) \rangle - \langle \nabla F(X(s-)), \Delta K(s) \rangle,$$

we obtain the equality (6), where we have used the finite variation property of the regulator $K = (K(t); t \geq 0)$ and the following equality:

$$\int_0^t \langle \nabla F(X(s-)), dK(s) \rangle - \sum_{0 < s \leq t} \langle \nabla F(X(s-)), \Delta K(s) \rangle = \int_0^t \langle \nabla F(X(s)), dK^c(s) \rangle, \quad t > 0.$$

Thus, we complete the proof of the lemma. \square

Corollary 2.1. *Let $\lambda \in \mathbb{R}$. Then, the d -dimensional strong solution process $X = (X(t); t \geq -\tau)$ to RSDDEJ (1) satisfies the following inequality:*

$$\begin{aligned} e^{\lambda t} |X(t)|^2 &\leq |\xi(0)|^2 + 2 \int_0^t e^{\lambda s} \langle X(s), b(s, X(s), X(s-\tau)) \rangle ds + \lambda \int_0^t e^{\lambda s} |X(s)|^2 ds \\ &\quad + \int_0^t e^{\lambda s} dM(s) + \int_0^t e^{\lambda s} \|\sigma(s, X(s), X(s-\tau))\|^2 ds \\ (11) \quad &\quad + \int_0^t \int_{\mathcal{E}} e^{\lambda s} |g(s, X(s), X(s-\tau), \rho)|^2 \nu(d\rho) ds, \end{aligned}$$

where the process $M = (M(t); t \geq 0)$ is defined by

$$\begin{aligned} M(t) &= 2 \int_0^t \langle X(s), \sigma(s, X(s), X(s-\tau)) dW(s) \rangle \\ &\quad + \int_0^t \int_{\mathcal{E}} [|[X(s-) + g(s, X(s-), X((s-\tau)-), \rho)]^+|^2 - |X(s-)|^2] \tilde{N}(d\rho, ds). \end{aligned}$$

Proof. For $x \in \mathbb{R}_+^d$, we take the smooth function $F(x) = |x|^2$ in Lemma 2.1. Then the corresponding equality (6) reads

$$\begin{aligned} |X(t)|^2 &= |\xi(0)|^2 + 2 \int_0^t \langle X(s), b(s, X(s), X(s-\tau)) \rangle ds + \int_0^t \|\sigma(s, X(s), X(s-\tau))\|^2 ds \\ &\quad + M(t) + \int_0^t \int_{\mathcal{E}} [|[X(s-) + g(s, X(s-), X((s-\tau)-), \rho)]^+|^2 - |X(s-)|^2 \\ &\quad - 2 \langle X(s-), g(s, X(s-), X((s-\tau)-), \rho) \rangle] \nu(d\rho) ds, \end{aligned}$$

where we used the support property (5), namely

$$\int_0^t \langle X(s), dK^c(s) \rangle = 0, \quad \forall t \geq 0.$$

For $(t, x, y, \rho) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathcal{E}$, applying the following inequality

$$(12) \quad |[x + g(t, x, y, \rho)]^+|^2 - |x|^2 - 2 \langle x, g(t, x, y, \rho) \rangle \leq |g(t, x, y, \rho)|^2,$$

and conclude the validity of the inequality (11) by using integration by parts. \square

Corollary 2.2. Let $H(\cdot) \in C^2(\mathbb{R}_+^d)$. Define the following function as

$$(13) \quad \begin{aligned} Q^H(t, x, y) &= \langle \nabla H(x), b(t, x, y) \rangle + \frac{1}{2} \text{Tr} [(\sigma \sigma^T)(t, x, y)(D^2 H(x) + \nabla H(x) \otimes \nabla H(x))] \\ &+ \int_{\mathcal{E}} [\exp \{H([x + g(t, x, y, \rho)]^+) - H(x)\} - 1 - \langle \nabla H(x), g(t, x, y, \rho) \rangle] \nu(d\rho), \end{aligned}$$

on $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$. We further define the positive process by

$$(14) \quad Y^E(t) := \exp \left(H(X(t)) - H(\xi(0)) - \int_0^t Q^H(s, X(s), X(s - \tau)) ds \right), \quad t \geq 0,$$

where $X = (X(t); t \geq -\tau)$ is the d -dimensional strong solution process to RSDDEJ (1). Then the positive process $Y^E = (Y^E(t); t \geq 0)$ satisfies the following equality:

$$(15) \quad Y^E(t) = 1 + M^E(t) + \int_0^t Y^E(s) \langle \nabla H(X(s)), dK^c(s) \rangle, \quad t \geq 0,$$

where the process $M^E = (M^E(t); t \geq 0)$ is an $(\mathcal{F}_t; t \geq 0)$ -local martingale taken values on \mathbb{R} .

Proof. For $x \in \mathbb{R}_+^d$, take the smooth function $F(x) = \exp(H(x))$ in Lemma 2.1. Using the representation of derivatives $\nabla F(x) = F(x) \nabla H(x)$ and $D^2 F(x) = F(x)[D^2 H(x) + \nabla H(x) \otimes \nabla H(x)]$, we arrive at

$$\frac{d \exp \{H(X(t)) - H(\xi(0))\}}{\exp \{H(X(t)) - H(\xi(0))\}} = Q^H(t, X(t), X(t - \tau)) dt + d\hat{M}(t) + \langle \nabla H(X(t)), dK^c(t) \rangle,$$

where $\hat{M} = (\hat{M}(t); t \geq 0)$ is a real-valued $(\mathcal{F}_t; t \geq 0)$ -local martingale. Then the equality (15) follows from applying integration by parts to $\exp \{H(X(t)) - H(\xi(0))\} \exp(-\int_0^t Q^H(s, X(s), X(s - \tau)) ds)$. \square

The Corollary 2.1 can be used to establish the estimate of the second-order moment for the d -dimensional segment process $(X_t; t \geq 0)$ corresponding to the d -dimensional solution process $X = (X(t); t \geq -\tau)$.

Proposition 2.1. Under the condition **(A1)**, it holds that

$$(16) \quad \sup_{t \geq 0} \mathbb{E} [\|X_t\|^2] < +\infty.$$

Proof. Let $\lambda > 0$. By virtue of the inequality (11) in Corollary 2.1, we arrive at, under the condition **(A1)**,

$$\begin{aligned} & \mathbb{E} [e^{\lambda t} |X(t)|^2] \\ & \leq \mathbb{E} [|\xi(0)|^2] + 2\mathbb{E} \left[\int_0^t e^{\lambda s} \langle X(s), b(s, X(s), X(s - \tau)) \rangle ds \right] \\ & \quad + \mathbb{E} \left[\lambda \int_0^t e^{\lambda s} |X(s)|^2 ds \right] + \mathbb{E} \left[\int_0^t e^{\lambda s} \|\sigma(s, X(s), X(s - \tau))\|^2 ds \right] \\ & \quad + \mathbb{E} \left[\int_0^t \int_{\mathcal{E}} e^{\lambda s} |g(s, X(s), X(s - \tau), \rho)|^2 \nu(d\rho) ds \right] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} [|\xi(0)|^2] + \alpha \int_0^t e^{\lambda s} ds + (\lambda - \alpha_1) \mathbb{E} \left[\int_0^t e^{\lambda s} |X(s)|^2 ds \right] + \alpha_2 \mathbb{E} \left[\int_0^t e^{\lambda s} |X(s - \tau)|^2 ds \right] \\
&= \mathbb{E} [|\xi(0)|^2] + \frac{\alpha}{\lambda} (e^{\lambda t} - 1) + e^{\lambda \tau} \mathbb{E} \left[\int_{-\tau}^0 e^{\lambda v} |\xi(v)|^2 dv \right] + (\lambda - \alpha_1 + \alpha_2 e^{\lambda \tau}) \mathbb{E} \left[\int_0^t e^{\lambda s} |X(s)|^2 ds \right].
\end{aligned}$$

Now we can choose a constant $\lambda^* > 0$ such that $\lambda^* - \alpha_1 + \alpha_2 e^{\lambda^* \tau} = 0$, since $\alpha_1 > \alpha_2$. Then, for all $t \geq 0$,

$$\begin{aligned}
\mathbb{E} [|X(t)|^2] &\leq e^{-\lambda^* t} \left\{ \frac{\alpha}{\lambda^*} (e^{\lambda^* t} - 1) + \mathbb{E} [|\xi(0)|^2] + e^{\lambda^* \tau} \mathbb{E} \left[\int_{-\tau}^0 e^{\lambda^* v} |\xi(v)|^2 dv \right] \right\} \\
(17) \quad &\leq \frac{\alpha}{\lambda^*} + \mathbb{E} [|\xi(0)|^2] + e^{\lambda^* \tau} \mathbb{E} \left[\int_{-\tau}^0 e^{\lambda^* v} |\xi(v)|^2 dv \right].
\end{aligned}$$

Let $\theta \in [-\tau, 0]$. For any $t > \tau$, it follows from Lemma 2.1 that

$$\begin{aligned}
|X(t + \theta)|^2 &= |X(t - \tau)|^2 + 2 \int_{t-\tau}^{t+\theta} \langle X(s), b(s, X(s), X(s - \tau)) \rangle ds \\
&\quad + \int_{t-\tau}^{t+\theta} \|\sigma(s, X(s), X(s - \tau))\|^2 ds \\
&\quad + 2 \int_{t-\tau}^{t+\theta} \langle X(s), \sigma(s, X(s), X(s - \tau)) dW(s) \rangle \\
&\quad + 2 \int_{t-\tau}^{t+\theta} \int_{\mathcal{E}} \langle X(s), g(s, X(s), X(s - \tau), \rho) \rangle \tilde{N}(d\rho, ds) \\
&\quad + \int_{t-\tau}^{t+\theta} \int_{\mathcal{E}} [|X(s-) + g(s, X(s-), X((s - \tau)-), \rho)|^2 - |X(s-)|^2 \\
&\quad - 2 \langle X(s-), g(s, X(s-), X((s - \tau)-), \rho) \rangle] N(d\rho, ds).
\end{aligned}$$

Using the Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}
&\mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} \left| \int_{t-\tau}^{t+\theta} \langle X(s), \sigma(s, X(s), X(s - \tau)) dW(s) \rangle \right| \right] \\
&\leq \frac{1}{2} \mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} |X(t + \theta)|^2 \right] + C \mathbb{E} \left[\int_{t-\tau}^t \|\sigma(s, X(s), X(s - \tau))\|^2 ds \right],
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} \left| \int_{t-\tau}^{t+\theta} \int_{\mathcal{E}} \langle X(s), g(s, X(s-), X((s - \tau)-), \rho) \rangle \tilde{N}(d\rho, ds) \right| \right] \\
&\leq \frac{1}{4} \mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} |X(t + \theta)|^2 \right] + C \mathbb{E} \left[\int_{t-\tau}^t \int_{\mathcal{E}} |g(s, X(s), X(s - \tau), \rho)|^2 \nu(d\rho) ds \right],
\end{aligned}$$

where $C > 0$ is some positive constant. On the other hand, by using the inequality (12), it follows that

$$\mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} \int_{t-\tau}^{t+\theta} \int_{\mathcal{E}} [|X(s-) + g(s, X(s-), X((s - \tau)-), \rho)|^2 - |X(s-)|^2] N(d\rho, ds) \right]$$

$$\begin{aligned}
& -2 \langle X(s-), g(s, X(s-), X((s-\tau)-), \rho) \rangle] N(d\rho, ds) \Big] \\
& \leq \mathbb{E} \left[\sup_{-\tau \leq \theta \leq 0} \int_{t-\tau}^{t+\theta} \int_{\mathcal{E}} |g(s, X(s-), X((s-\tau)-), \rho)|^2 N(d\rho, ds) \right] \\
& \leq \mathbb{E} \left[\int_{t-\tau}^t \int_{\mathcal{E}} |g(s, X(s), X(s-\tau), \rho)|^2 \nu(d\rho) ds \right].
\end{aligned}$$

Hence, it holds that

$$(18) \quad \mathbb{E} [\|X_t\|^2] \leq 4\mathbb{E} [|X(t-\tau)|^2] + C \int_{t-\tau}^t \mathbb{E} [|X(s-\tau)|^2] ds,$$

for some positive constant C which is independent of time t . The required assertion follows from (17). \square

The following result relates to the exponential moment of the d -dimensional solution process X , which is a reflected delay version of Röchner and Zhang [RZ07]'s exponential integrability of the solution without reflection and delay when the drift and diffusion coefficients (b, σ) are uniformly bounded on $(t, x, y) \in [0, T] \times \mathbb{R}_+^d \times \mathbb{R}_+^d$ with $T > 0$. In particular, we can establish the exponential moment estimate of the following RSDDEJ without drift and diffusive parts:

$$\begin{aligned}
X(t) &= \xi(0) + \int_0^t \int_{\mathcal{E}} g(s, X(s-), X((s-\tau)-), \rho) \tilde{N}(d\rho, ds) + K(t) \\
(19) \quad &=: \xi(0) + \tilde{Y}(t) + K(t), \quad t \geq 0,
\end{aligned}$$

where the d -dimensional process $\tilde{Y} = (\tilde{Y}^i(t); t \geq 0)_{d \times 1}$ is the compensated version of the pure jump process $Y = (Y^i(t); t \geq 0)_{d \times 1}$ defined as (8).

Lemma 2.2. *For the characteristic measure and the jump coefficient (ν, g) , suppose that there exists a constant $\ell_g > 0$ such that*

$$(20) \quad |g(t, x, y, \rho)| \leq \ell_g h(t, \rho), \quad \forall (t, x, y, \rho) \in [0, T] \times \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathcal{E},$$

where $T > 0$ and $h(t, \rho)$ is a nonnegative measurable function satisfying

$$(21) \quad \sup_{0 \leq t \leq T} \int_{\mathcal{E}} h^2(t, \rho) e^{h(t, \rho)} \nu(d\rho) < +\infty,$$

for any finite $\varrho > 0$. If the drift and diffusion coefficients (b, σ) are uniformly bounded on $(t, x, y) \in [0, T] \times \mathbb{R}_+^d \times \mathbb{R}_+^d$, then

$$(22) \quad \mathbb{E} \left[\exp \left(a \sup_{0 \leq t \leq T} |X(t)| \right) \right] < +\infty,$$

for any finite $a > 0$.

Proof. We adopt the test function used in Röchner and Zhang [RZ07] to discuss our reflected delay case. Consider the following smooth function on \mathbb{R}_+^d ,

$$(23) \quad H_\lambda(x) = \sqrt{1 + \lambda|x|^2}, \quad \lambda > 0, \text{ and } x \in \mathbb{R}_+^d.$$

Then, the gradient $\nabla H_\lambda(x) = \lambda H_\lambda^{-1}(x)x$ with $x \in \mathbb{R}_+^d$. For any $i, j = 1, 2, \dots, d$ and $x \in \mathbb{R}_+^d$, the partial derivatives

$$(24) \quad \frac{\partial H_\lambda(x)}{\partial x_i} \leq \sqrt{\lambda}, \quad \text{and} \quad \frac{\partial^2 H_\lambda(x)}{\partial x_i \partial x_j} + \frac{\partial H_\lambda(x)}{\partial x_i} \frac{\partial H_\lambda(x)}{\partial x_j} \leq 2\lambda.$$

Recall the real-valued process $Y^E = (Y^E(t); t \geq 0)$ given by (15). Note that, for all $t \geq 0$,

$$\begin{aligned} 0 \leq \int_0^t \langle \nabla H_\lambda(X(s)), dK^c(s) \rangle &= \int_0^t \lambda H_\lambda^{-1}(X(s)) \langle X(s), dK^c(s) \rangle \\ &\leq \lambda \int_0^t \langle X(s), dK^c(s) \rangle = 0, \end{aligned}$$

by employing the support property (5). This implies that

$$\int_0^t \langle \nabla H_\lambda(X(s)), dK^c(s) \rangle = 0, \quad \forall t \geq 0.$$

Then by Corollary 2.2, we have that

$$Y_\lambda^E(t) := \exp \left(H_\lambda(X(t)) - H_\lambda(\xi(0)) - \int_0^t Q^{H_\lambda}(s, X(s), X(s-\tau)) ds \right), \quad t \geq 0$$

is a positive $(\mathcal{F}_t; t \geq 0)$ -local martingale and hence it is a supermartingale, where the function $Q_\lambda^H(t, x, y)$ is given by (13) with the function $H(x)$ replaced by $H_\lambda(x)$. Using the inequality $H_\lambda([x + g(t, x, y, \rho)]^+) \leq H_\lambda(x + g(t, x, y, \rho))$ and the estimates of derivatives (24), it is not difficult to prove that, for any $(t, x, y) \in [0, T] \times \mathbb{R}_+^d \times \mathbb{R}_+^d$,

$$Q^{H_\lambda}(t, x, y) \leq C_1 \left[1 + \sup_{0 \leq t \leq T} \int_{\mathcal{E}} h^2(t, \rho) e^{\sqrt{\lambda} h(t, \rho)} \nu(d\rho) \right] := C_2 < +\infty,$$

under the conditions (20) and (21), where positive constants $C_1 = C_1(d, \lambda)$ and $C_2 = C_2(d, \lambda, T)$ depend on the dimension number d , the parameter λ and the time level T only. Based on the above estimate of the function $Q^{H_\lambda}(t, x, y)$, the desired result follows from Proposition 4.2 and Corollary 4.3 in Röchner and Zhang [RZ07]. \square

3 Invariant measures

In this section, we will establish the existence and uniqueness of invariant measures of the d -dimensional segment process $X_t(\theta) = X(t + \theta)$ with $-\tau \leq \theta \leq 0$ under the condition **(A1)** and **(A2)**.

For the initial data $\xi \in D([-\tau, 0]; \mathbb{R}_+^d)$, we use $(X_t^\xi; t \geq 0)$ to represent the corresponding segment process to the d -dimensional solution process $X = (X(t); t \geq -\tau)$ with $X(t) = \xi(t)$ on $[-\tau, 0]$. Define the Markov semigroup associated with the segment process by

$$(25) \quad \mathcal{P}_t f(\xi) = \mathbb{E} \left[f \left(X_t^\xi \right) \right],$$

for all bounded continuous functions $f(\cdot)$ defined on the space $D([-\tau, 0]; \mathbb{R}_+^d)$. For any finite time level $T > 0$, define the probability measure $Q_T(\xi; \cdot)$ by

$$(26) \quad Q_T(\xi; A) = \frac{1}{T} \int_0^T \mathcal{P}_t(\xi; A) dt, \quad \text{for } A \in \mathcal{M},$$

where $\mathcal{P}_t(\xi; A) := \mathcal{P}_t \mathbb{1}_A(\xi)$ and \mathcal{M} corresponds to the Borel σ -algebra generated by the space $D([-\tau, 0]; \mathbb{R}_+^d)$.

Recall the d -dimensional compensated pure jump process $\tilde{Y} = (\tilde{Y}^i(t); t \geq 0)_{d \times 1}$ defined as (19). We have the following proposition.

Proposition 3.1. *Suppose that the jump coefficient g satisfies the following linear growth-type condition:*

$$(27) \quad |g(t, x, y, \rho)| \leq \ell_g(1 + |x| + |y|)h(t, \rho), \quad \text{for all } (t, x, y, \rho) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d \times \mathcal{E},$$

where $h(t, \rho)$ is a nonnegative measurable function satisfying

$$(28) \quad \sup_{t \geq 0} \int_{\mathcal{E}} h^2(t, \rho) e^{\varrho h(t, \rho)} \nu(d\rho) < +\infty,$$

for any finite $\varrho > 0$. Let $u \geq \tau$. Then, for any $\varepsilon > 0$ and $r > 0$, there exists a constant $\delta_{\varepsilon, r} > 0$ such that whenever $\delta \in (0, \delta_{\varepsilon, r}]$,

$$(29) \quad \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \geq r \right) < \varepsilon.$$

Proof. Let $m > 0$. Define the following $(\mathcal{F}_t; t \geq 0)$ -stopping time by

$$\tau_m = \inf \left\{ t \geq 0; \|X\|_{[-\tau, t]} > m \right\},$$

Then, $\tau_m \rightarrow \infty$ almost surely as $m \rightarrow \infty$ due to Proposition 2.1. Thus, we can choose a constant $m > 0$ large enough so that

$$(30) \quad \begin{aligned} & \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \geq r \right) \\ & \leq \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \geq r, \tau_m > u \right) + \mathbb{P}(\tau_m \leq u) \\ & \leq \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}((u+t) \wedge \tau_m) - \tilde{Y}((u+s) \wedge \tau_m)| \geq r \right) + \frac{\varepsilon}{2}. \end{aligned}$$

It remains to prove that there exists a constant $\delta_{\varepsilon, r} > 0$ such that whenever $\delta \in (0, \delta_{\varepsilon, r}]$,

$$(31) \quad \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}((u+t) \wedge \tau_m) - \tilde{Y}((u+s) \wedge \tau_m)| \geq r \right) < \frac{\varepsilon}{2}.$$

Using the condition (27), the set

$$\{\tau_m > s\} \subset \{g(s \wedge \tau_m, X((s \wedge \tau_m)), X((s - \tau) \wedge \tau_m), \rho) \leq \ell_g(1 + 2m)h(s \wedge \tau_m, \rho)\},$$

for any $s > 0$. Accordingly, to prove (31), it suffices to check that for any $\varepsilon > 0$ and $r > 0$, there exists a constant $\delta_{\varepsilon,r} > 0$ such that whenever $\delta \in (0, \delta_{\varepsilon,r}]$,

$$(32) \quad \mathbb{P} \left(\sup_{s,t \in [-\tau, 0], |s-t| < \delta} \left| \tilde{Y}(u+t) - \tilde{Y}(u+s) \right| \geq r \right) < \frac{\varepsilon}{2},$$

under the conditions (20) and (28).

We next take the smooth function $H_\lambda(\cdot)$ on \mathbb{R}_+^d given by (23). Then, by Lemma 2.1, the process

$$Y_\lambda^E(t) := \exp \left(H_\lambda(\tilde{Y}(t)) - 1 - \int_0^t \tilde{Q}^{H_\lambda}(s, X(s), X(s-\tau)) ds \right), \quad t \geq 0$$

is a positive $(\mathcal{F}_t; t \geq 0)$ -local martingale, where the function $\tilde{Q}^{H_\lambda}(t, x, y)$ with $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$ is given by

$$\tilde{Q}^{H_\lambda}(t, x, y) = \int_{\mathcal{E}} [\exp \{ H_\lambda([x + g(t, x, y, \rho)]^+) - H_\lambda(x) \} - 1 - \langle \nabla H_\lambda(x), g(t, x, y, \rho) \rangle] \nu(d\rho).$$

Moreover, with the help of the conditions (20) and (28), we have $\tilde{Q}^{H_\lambda}(t, x, y) \leq \lambda C$ for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$, where $C = C(d) > 0$ is some constant depending on the dimension number d . Let $u \geq \tau$ and $t \in [-\tau, 0]$. For any $r, \lambda > 0$, we have

$$\begin{aligned} & \mathbb{P} \left(\frac{|\tilde{Y}(u+t)|}{\sqrt{|u+t|}} \geq r \right) \\ &= \mathbb{P} \left(H_\lambda(\tilde{Y}(u+t)) \geq \sqrt{1 + \lambda r^2 |u+t|} \right) \\ &\leq \mathbb{P} \left(1 + \int_0^{u+t} \tilde{Q}^{H_\lambda}(s, X(s), X(s-\tau)) ds + \log(Y_\lambda^E(u+t)) \geq \sqrt{1 + \lambda r^2 |u+t|} \right) \\ &\leq \exp \left[1 + \lambda(u+t)C - \sqrt{1 + \lambda r^2 |u+t|} \right] \mathbb{E} [\tilde{Y}(u+t)] \\ &\leq \exp \left[1 + \lambda(u+t)C - \sqrt{\lambda r^2 |u+t|} \right], \end{aligned}$$

where the constant $C = C(d) > 0$ depends on the dimensional number d only. Taking the parameter $\lambda = \beta|u+t|^{-1}r^2$ for some $\beta > 0$. Then

$$\mathbb{P} \left(\frac{|\tilde{Y}(u+t)|}{\sqrt{|u+t|}} \geq r \right) \leq \exp \left(-r^2 \sqrt{\beta} + 1 + r^2 \beta C \right).$$

Choose a constant $\beta > 0$ small enough so that $\beta^* := \beta(\beta^{-\frac{1}{2}} - C) > 0$. Then it holds that

$$\mathbb{P} \left(\frac{|\tilde{Y}(u+t)|}{\sqrt{|u+t|}} \geq r \right) \leq \exp \left(-\beta^* r^2 + 1 \right).$$

Using integration by parts, for any $\beta_0 \in (0, \beta^*]$, we have

$$\mathbb{E} \left[\beta_0 \frac{|\tilde{Y}(u+t)|^2}{|u+t|} \right] < \infty,$$

which yields that there exists a constant $C > 0$ such that

$$(33) \quad R := \sup_{s, t \in [-\tau, 0], s \neq t} \mathbb{E} \left[\exp \left(C \frac{|\tilde{Y}(u+t) - \tilde{Y}(u+s)|^2}{|s-t|} \right) \right] < \infty.$$

Define the following positive valued positive random variable as

$$V = \int_{u-\tau}^u \int_{u-\tau}^u \exp \left(C \frac{|\tilde{Y}(v_1) - \tilde{Y}(v_2)|^2}{|v_1 - v_2|} \right) dv_1 dv_2.$$

Then $\mathbb{E}[V] \leq \tau^2 R < \infty$. From Garsia-Rodemich-Rumsey's Lemma (see [SV79, Theorem 2.1.3, p47]), it follows that

$$|\tilde{Y}(u+t) - \tilde{Y}(u+s)| \leq C \int_0^{|s-t|} \sqrt{\log \left(\frac{V}{v^2} \right)} dv.$$

Thus, there exists a constant $\kappa > 0$ small enough such that

$$(34) \quad |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \leq C \left[1 + \sqrt{\log(V)} \right] |t-s|^\kappa.$$

We arrive at

$$\begin{aligned} & \mathbb{P} \left(\sup_{s, t \in [-\tau, 0], |s-t| < \delta} |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \geq r \right) \\ & \leq \mathbb{P} \left(C \left[1 + \sqrt{\log(V)} \right] \delta^\kappa \geq r \right) \leq \mathbb{P} \left(V \geq \exp \left[|rC^{-1}\delta^{-\kappa} - 1|^2 \right] \right) \\ & \leq \mathbb{E}[V] \exp \left[-|rC^{-1}\delta^{-\kappa} - 1|^2 \right] \leq \tau^2 R \exp \left[-|rC^{-1}\delta^{-\kappa} - 1|^2 \right], \end{aligned}$$

which yields (32). □

Remark 3.1. Let $t \geq 0$. Define d -dimensional $(\mathcal{F}_t; t \geq 0)$ -local martingale $Z(t) = \int_0^t \sigma(s, X(s), X(s-\tau)) dW(s)$. Kinnally and Williams [KW10] proved that the similar conclusion of Proposition 3.1 holds if the diffusion coefficient σ satisfies the linear growth condition. The similar result see (23) in [KW10].

Remark 3.2. Let the conditions of Proposition 3.1 hold. Then, for any $\varepsilon > 0$ and $r > 0$, there exists a constant $\delta_{\varepsilon, r} > 0$ such that whenever $\delta \in (0, \delta_{\varepsilon, r}]$,

$$(35) \quad \mathbb{P} \left(\sup_{(t_1 < t_2 \in [-\tau, 0]; |t_1 - t_2| < \delta)} \sup_{t \in [t_1, t_2]} [|U(u+t) - U(u+t_1)| \wedge |U(u+t) - U(u+t_2)|] \geq r \right) < \varepsilon,$$

where $U \in \{Z, \tilde{Y}\}$. Indeed, using similar arguments to that of Proposition 3.1, it suffices to prove the validity of (35) with $U = \tilde{Y}$ under the conditions (20) and (28). Note that

$$\sup_{(t_1 < t_2 \in [-\tau, 0]; |t_1 - t_2| < \delta)} \sup_{t \in [t_1, t_2]} \left[|\tilde{Y}(u+t) - \tilde{Y}(u+t_1)| \wedge |\tilde{Y}(u+t) - \tilde{Y}(u+t_2)| \right]$$

$$(36) \quad \leq 2 \sup_{(t_1 < t_2 \in [-\tau, 0]; |t_1 - t_2| < \delta)} \sup_{t \in [t_1, t_2]} \left| \tilde{Y}(u+t) - \tilde{Y}(u+t_1) \right|.$$

Let $t_1 < t_2 \in [-\tau, 0]$ be fixed and satisfies $|t_2 - t_1| < \delta$. Using (34), for all $t \in [t_1, t_2]$, we have

$$\left| \tilde{Y}(u+t) - \tilde{Y}(u+t_1) \right| \leq C \left[1 + \sqrt{\log(V)} \right] |t - t_1|^\kappa \leq C \left[1 + \sqrt{\log(V)} \right] \delta^\kappa,$$

which implies that

$$\sup_{(t_1 < t_2 \in [-\tau, 0]; |t_1 - t_2| < \delta)} \sup_{t \in [t_1, t_2]} \left| \tilde{Y}(u+t) - \tilde{Y}(u+t_1) \right| \leq C \left[1 + \sqrt{\log(V)} \right] \delta^\kappa.$$

The remaining proof of (35) is very similar to that of Proposition 3.1.

Based on the above auxiliary results concerning stochastic integrals with respect to compensated Poisson measure \tilde{N} and n -dimensional Brownian motion W , we have the following main result of this paper.

Theorem 3.1. *Let the conditions (A1) and (A2) hold. Suppose that the jump coefficient g and characteristic measure ν satisfy (27) and (28), respectively. The drift coefficient b satisfy the growth condition: $|b(t, x, y)| \leq \ell_b(1 + |x|^k + |y|^k)$ with $k = 1$ or 2 , for all $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d$, where $\ell_b > 0$. Then, the Markov semigroup $(\mathcal{P}_t; t \geq 0)$ defined as (25) for the d -dimensional segment process to RSDDEJ (1) admits a unique invariant measure.*

Proof. We first deal with the existence of an invariant measure for the Markov semigroup $(\mathcal{P}_t; t \geq 0)$ defined as (25). Let $(t_n; n \in \mathbb{N})$ be a sequence of times increasing to $+\infty$. We prove that the sequence of probability measure $(Q_{t_n}(\xi; \cdot); n \in \mathbb{N})$ is tight on the Skorohod space $(D([-\tau, 0]; \mathbb{R}_+^d), \mathcal{M})$. Since the Markov transition semigroup $(\mathcal{P}_t(\xi; \cdot); t \geq t_0)$ is Fellerian for some time $t_0 \geq 0$ (see Section 3.3 in Reib, et al. [RRG06]), any weak limit point is an invariant measure by virtue of Krylov-Bogulyubov's Theorem (see e.g., [DZ96] and [PRG06]). Now fix a finite time level $T > 0$. Then, for any $r > 0$,

$$(37) \quad \begin{aligned} Q_T \left(\eta(\cdot) \in D([-\tau, 0]; \mathbb{R}_+^d); |\eta(0)| > r \right) &= \frac{1}{T} \int_0^T \mathbb{P} \left(\left| X^\xi(t) \right| > r \right) dt \\ &\leq \frac{1}{r^2} \sup_{t \geq 0} \mathbb{E} \left[\left| X^\xi(t) \right|^2 \right], \end{aligned}$$

which tends to zero when $r \rightarrow \infty$ by employing Proposition 2.1. Let $\delta > 0$. Define the modulus of continuity of any function $\eta(\cdot) \in D(I; \mathbb{R}_+^d)$ with a subinterval I of \mathbb{R} :

$$\begin{aligned} w(\eta; \delta) &:= \sup_{(s, t \in I; |s-t| < \delta)} |\eta(s) - \eta(t)|, \quad \text{and} \\ w^*(\eta; \delta) &:= \sup_{(t_1 < t_2 \in I; t_2 - t_1 < \delta)} \sup_{t_1 \leq t \leq t_2} [|\eta(t) - \eta(t_1)| \wedge |\eta(t) - \eta(t_2)|]. \end{aligned}$$

Using the solution representation of Skorohod problem (see e.g., [A03]), the regulator $K = (K^i(t); t \geq 0)_{d \times 1}$ admits the representation:

$$(38) \quad K(t) = \sup_{0 \leq s \leq t} [\Gamma(X)(s)]^-, \quad t \geq 0,$$

where the d -dimensional process $\Gamma(X) = (\Gamma(X)(t); t \geq 0)$ is defined as

$$(39) \quad \begin{aligned} \Gamma(X)(t) &= \xi(0) + \int_0^t b(s, X(s), X(s-\tau))ds + \int_0^t \sigma(s, X(s), X(s-\tau))dW(s) \\ &+ \int_0^t \int_{\mathcal{E}} g(s, X(s-), X((s-\tau)-), \rho) \tilde{N}(d\rho, ds). \end{aligned}$$

Then, for any $u \geq \tau$, we arrive at

$$\begin{aligned} & \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |X(u+t) - X(u+s)| \\ = & \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |\Gamma(X)(u+t) - \Gamma(X)(u+s) + K(u+t) - K(u+s)| \\ \leq & 2 \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |\Gamma(X)(u+t) - \Gamma(X)(u+s)| \\ \leq & 2 \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} \int_{u+s}^{u+t} |b(v, X(v), X(v-\tau))| dv + 2 \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |Z(u+t) - Z(u+s)| \\ & + 2 \sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |\tilde{Y}(u+t) - \tilde{Y}(u+s)|, \end{aligned}$$

where the stochastic integral processes $Z = (Z(t); t \geq 0)$ and $\tilde{Y} = (\tilde{Y}; t \geq 0)$ are defined as in Remark 3.1 and Remark 3.2 respectively. Hence, for any $r > 0$ and $k = 1$ or 2 ,

$$\begin{aligned} \mathbb{P}(w(X_u; \delta) \geq r) &\leq \mathbb{P}\left(\sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} \int_{u+s}^{u+t} [1 + |X(v)|^k + |X(v-\tau)|^k] dv \geq \frac{r}{6\ell_b}\right) \\ &+ \mathbb{P}\left(\sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |Z(u+t) - Z(u+s)| \geq \frac{r}{6}\right) \\ &+ \mathbb{P}\left(\sup_{(s < t \in [-\tau, 0]; |s-t| < \delta)} |\tilde{Y}(u+t) - \tilde{Y}(u+s)| \geq \frac{r}{6}\right) \\ (40) \quad &=: F_1(u, \delta; r) + F_2(u, \delta; r) + F_3(u, \delta; r). \end{aligned}$$

Take the following inequalities into account

$$\begin{aligned} \sup_{u \geq \tau} \mathbb{P}\left(\|X\|_{[u-2\tau, u]}^k > r\right) &\leq \begin{cases} \frac{1}{r^2} \sup_{t \geq 0} \mathbb{E}[\|X_t\|^2], & k = 1, \\ \frac{1}{r} \sup_{t \geq 0} \mathbb{E}[\|X_t\|^2], & k = 2, \end{cases} \\ F_1(u, \delta; r) &\leq \mathbb{P}\left(\delta[C_1 + C_2\|X\|_{[u-2\tau, u]}^k] \geq \frac{r}{6\ell_b}\right), \quad \text{for } k = 1, \text{ or } 2. \end{aligned}$$

By virtue of (16) in Proposition 2.1, for any $\varepsilon, r > 0$, there exists a constant $\delta_{\varepsilon, r}^1 > 0$ such that $\sup_{u \geq \tau} F_1(u, \delta; r) < \frac{\varepsilon}{6}$ whenever $\delta \in (0, \delta_{\varepsilon, r}^1]$. It follows from Proposition 3.1 and Remark 3.1 that there exist constants $\delta_{\varepsilon, r}^2 > 0$ and $\delta_{\varepsilon, r}^3 > 0$ such that $\sup_{u \geq \tau} F_2(u, \delta; r) < \frac{\varepsilon}{6}$ when $\delta \in (0, \delta_{\varepsilon, r}^2]$ and $\sup_{u \geq \tau} F_3(u, \delta; r) < \frac{\varepsilon}{6}$ for $\delta \in (0, \delta_{\varepsilon, r}^3]$ respectively. Finally, we obtain

$$(41) \quad \sup_{u \geq \tau} \mathbb{P}(w(X_u; \delta) \geq r) < \frac{\varepsilon}{2},$$

whenever $\delta \in (0, \wedge_{i=1}^3 \delta_{\varepsilon, r}^i]$. For the above δ , we further conclude that for all $T > \frac{2\tau}{\varepsilon} \vee \tau$,

$$(42) \quad Q_T \left(\eta(\cdot) \in D([- \tau, 0]; \mathbb{R}_+^d); w(\eta; \delta) \geq r \right) \leq \frac{\tau}{T} + \frac{1}{T} \int_{\tau}^T \mathbb{P}(w(X_u; \delta) \geq r) du < \varepsilon.$$

For any $\varepsilon, r > 0$, we can conclude that there exists a constant $\delta_{\varepsilon, r}^* > 0$ so that for all $\delta \in (0, \delta_{\varepsilon, r}^*]$, $Q_T(\eta(\cdot) \in D([- \tau, 0]; \mathbb{R}_+^d); w^*(\eta; \delta) \geq r) < \varepsilon$, by employing Remark 3.2. Together with (37) and (42), the sequence of probability measure $(Q_{t_n}(\xi; \cdot); n \in \mathbb{N})$ is tight on the Skorohod space $(D([- \tau, 0]; \mathbb{R}_+^d), \mathcal{M})$ in terms of Theorem 6.6 in Liptser and Shirayev [LS86].

Next we check the uniqueness of invariant measures under the condition **(A2)**. Let $X^\xi = (X^\xi(t); t \geq -\tau)$ and $X^\eta = (X^\eta(t); t \geq -\tau)$ be two strong solutions to RSDDEJ (1) with respect initial datum $\xi(\cdot)$ and $\eta(\cdot)$ on the space $D([- \tau, 0]; \mathbb{R}_+^d)$. Then, for $t \geq 0$,

$$\begin{aligned} X^\xi(t) - X^\eta(t) &= \xi(0) - \eta(0) + \int_0^t [b(s, X^\xi(s), X^\xi(s-\tau)) - b(s, X^\eta(s), X^\eta(s-\tau))] ds \\ &\quad + \int_0^t [\sigma(s, X^\xi(s), X^\xi(s-\tau)) - \sigma(s, X^\eta(s), X^\eta(s-\tau))] dW(s) \\ &\quad + \int_0^t \int_{\mathcal{E}} [g(s, X^\xi(s-), X^\xi((s-\tau)-), \rho) - g(s, X^\eta(s-), X^\eta((s-\tau)-), \rho)] \tilde{N}(d\rho, ds) \\ &\quad + K^\xi(t) - K^\eta(t), \end{aligned}$$

where $K^\xi = (K^\xi(t); t \geq 0)$ and $K^\eta = (K^\eta(t); t \geq 0)$ denotes the regulators for the solutions X^ξ and X^η respectively. For convenience, we let $b^j(t) := b(t, X^j(t), X^j(t-\tau))$, $\sigma^j(t) := \sigma(t, X^j(t), X^j(t-\tau))$ and $g^j(t, \rho) := g(t, X^j(t), X^j(t-\tau), \rho)$ with $j \in \{\xi, \eta\}$. Using Itô's formula (7), we arrive at

$$\begin{aligned} |X^\xi(t) - X^\eta(t)|^2 &= |\xi(0) - \eta(0)|^2 + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), (b^\xi - b^\eta)(s) \right\rangle ds \\ &\quad + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), (\sigma^\xi - \sigma^\eta)(s) dW(s) \right\rangle + \int_0^t \|(\sigma^\xi - \sigma^\eta)(s)\|^2 ds \\ &\quad + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), dK^{\xi, c}(s) - dK^{\eta, c}(s) \right\rangle \\ &\quad + \sum_{0 < s \leq t} \left[|(X^\xi - X^\eta)(s-)|^2 - |(X^\xi - X^\eta)(s-)|^2 \right. \\ &\quad \left. - 2 \left\langle (X^\xi - X^\eta)(s-), \Delta(Y^\xi - Y^\eta)(s) \right\rangle \right]. \end{aligned} \quad (43)$$

where the pure jump processes $Y^j(t) = \int_0^t \int_{\mathcal{E}} g^j(s-, \rho) N(d\rho, ds)$ and $K^{j, c}(t)$ corresponds to the continuous counterpart of $K^j(t)$ with $j \in \{\xi, \eta\}$. It follows from (10) that

$$\Delta(X^\xi - X^\eta)(t) = \varphi(X^\xi(t-), \Delta Y^\xi(t)) - \varphi(X^\xi(t-), \Delta Y^\eta(t)),$$

where the function $\varphi(x, y)$ is defined in (10) with $x, y \in \mathbb{R}_+^d$. Then

$$\begin{aligned} (X^\xi - X^\eta)(t-) + \Delta(X^\xi - X^\eta)(t) &= [X^\xi(t-) + \varphi(X^\xi(t-), \Delta Y^\xi(t))] \\ &\quad - [X^\eta(t-) + \varphi(X^\eta(t-), \Delta Y^\eta(t))] \\ &= [X^\xi(t-) + \Delta Y^\xi(t)]^+ - [X^\eta(t-) + \Delta Y^\eta(t)]^+. \end{aligned}$$

Accordingly, the equality (43) becomes

$$\begin{aligned}
\left|X^\xi(t) - X^\eta(t)\right|^2 &= |\xi(0) - \eta(0)|^2 + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), (b^\xi - b^\eta)(s) \right\rangle ds \\
&\quad + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), (\sigma^\xi - \sigma^\eta)(s) dW(s) \right\rangle + \int_0^t \|(\sigma^\xi - \sigma^\eta)(s)\|^2 ds \\
&\quad + 2 \int_0^t \left\langle (X^\xi - X^\eta)(s), dK^{\xi,c}(s) - dK^{\eta,c}(s) \right\rangle \\
&\quad + \int_0^t \int_{\mathcal{E}} \left[\left| [X^\xi(s-) + g^\xi(s-, \rho)]^+ - [X^\eta(s-) + g^\eta(s-, \rho)]^+ \right|^2 - |(X^\xi - X^\eta)(s-)|^2 \right. \\
&\quad \left. - 2 \left\langle (X^\xi - X^\eta)(s-), (g^\xi - g^\eta)(s-, \rho) \right\rangle \right] N(d\rho, ds).
\end{aligned}$$

Since the function $x \rightarrow [x]^+$ is Lipschitzian continuous, it holds that

$$\left| [X^\xi(t-) + g^\xi(t, \rho)]^+ - [X^\eta(t-) + g^\eta(t, \rho)]^+ \right|^2 \leq \left| (X^\xi - X^\eta)(t-) + (g^\xi - g^\eta)(t, \rho) \right|^2.$$

Let $\lambda > 0$. Take the condition **(A2)** into account, we have

$$\begin{aligned}
\mathbb{E} \left[e^{\lambda t} \left| X^\xi(t) - X^\eta(t) \right|^2 \right] &\leq \mathbb{E} \left[|\xi(0) - \eta(0)|^2 \right] + 2\mathbb{E} \left[\int_0^t e^{\lambda s} \left\langle (X^\xi - X^\eta)(s), (b^\xi - b^\eta)(s) \right\rangle ds \right] \\
&\quad + \mathbb{E} \left[\int_0^t e^{\lambda s} \|(\sigma^\xi - \sigma^\eta)(s)\|^2 ds \right] + \lambda \mathbb{E} \left[\int_0^t e^{\lambda s} \left| X^\xi(s) - X^\eta(s) \right|^2 ds \right] \\
&\quad + \mathbb{E} \left[\int_0^t \int_{\mathcal{E}} e^{\lambda s} |(g^\xi - g^\eta)(s, \rho)|^2 \nu(d\rho) ds \right] \\
&\leq \mathbb{E} \left[|\xi(0) - \eta(0)|^2 \right] + e^{\lambda \tau} \mathbb{E} \left[\int_{-\tau}^0 e^{\lambda v} |\xi(v) - \eta(v)|^2 dv \right] \\
&\quad + (\lambda - \beta_1 + \beta_2 e^{\lambda \tau}) \mathbb{E} \left[\int_0^t e^{\lambda s} |X^\xi - X^\eta(s)|^2 ds \right],
\end{aligned}$$

where we also used the following estimate concerning the regulators K^ξ and K^η , for $t \geq 0$,

$$\begin{aligned}
&\int_0^t e^{\lambda s} \left\langle (X^\xi - X^\eta)(s), dK^{\xi,c}(s) - dK^{\eta,c}(s) \right\rangle \\
&= \int_0^t e^{\lambda s} \left\langle X^\xi(s), dK^{\xi,c}(s) \right\rangle - \int_0^t e^{\lambda s} \left\langle X^\eta(s), dK^{\xi,c}(s) \right\rangle - \int_0^t e^{\lambda s} \left\langle X^\xi(s), dK^{\eta,c}(s) \right\rangle \\
&\quad + \int_0^t e^{\lambda s} \left\langle X^\eta(s), dK^{\eta,c}(s) \right\rangle \\
&= - \int_0^t e^{\lambda s} \left\langle X^\eta(s), dK^{\xi,c}(s) \right\rangle - \int_0^t e^{\lambda s} \left\langle X^\xi(s), dK^{\eta,c}(s) \right\rangle \\
&\leq 0,
\end{aligned}$$

where we have used the fact that, for $j \in \{\xi, \eta\}$, it holds that

$$0 \leq \int_0^t e^{\lambda s} \left\langle X^j, dK^{j,c}(s) \right\rangle \leq e^{\lambda t} \int_0^t \left\langle X^j, dK^{j,c}(s) \right\rangle = 0$$

for any finite time $t > 0$, by using the support property (5).

Recall the positive constant λ^* satisfying $\lambda^* - \beta_1 + \beta_2 e^{\lambda^* \tau} = 0$. Then, for all $t \geq 0$,

$$\mathbb{E} \left[\left| (X^\xi - X^\eta)(t) \right|^2 \right] \leq e^{-\lambda^* t} \left\{ \mathbb{E} [|\xi(0) - \eta(0)|^2] + e^{\lambda^* \tau} \mathbb{E} \left[\int_{-\tau}^0 e^{\lambda^* v} |\xi(v) - \eta(v)|^2 dv \right] \right\},$$

which shows that

$$(44) \quad \lim_{t \rightarrow +\infty} \mathbb{E} \left[\left| (X^\xi - X^\eta)(t) \right|^2 \right] = 0.$$

Using the similar proof to that of the inequality (18), we have

$$\mathbb{E} \left[\left\| X_t^\xi - X_t^\eta \right\|^2 \right] \leq 4\mathbb{E} \left[\left| (X^\xi - X^\eta)(t - \tau) \right|^2 \right] + C \int_{t-\tau}^t \mathbb{E} \left[\left| (X^\xi - X^\eta)(s - \tau) \right|^2 \right] ds,$$

for some constant $C > 0$ which is independent of time t . The above estimate and Gronwall's Lemma lead to

$$\lim_{t \rightarrow +\infty} \mathbb{E} \left[\left\| X_t^\xi - X_t^\eta \right\|^2 \right] = 0.$$

Thus, we complete the proof of the uniqueness of invariant measures. \square

4 Local time of the solution

As we have pointed out in section 1, the sizes of the jumps for the regulator $K = (K^i(t); t \geq 0)_{d \times 1}$ can be identified by (4). In this section, we will establish a relationship between the continuous counterpart of the regulator K and the local time of the solution process $X = (X(t); t \geq -\tau)$.

For $i = 1, 2, \dots, d$, let $L^i = (L^i(t); t \geq 0)$ be the local time process for the i th-element of the strong solution process X at point 0. Moreover, if $\sum_{0 < s \leq t} |\Delta Y^i(s)| < +\infty$ for all $t > 0$ (the pure jump process Y^i is defined as (8)), then the local time has the following limit representation by Protter [P04],

$$(45) \quad \begin{aligned} L^i(t) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[0, \varepsilon)}(X^i(s)) d \langle X^{i,c}, X^{i,c} \rangle_s \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_{j=1}^n \int_0^t \mathbb{1}_{[0, \varepsilon)}(X^i(s)) \sigma_{ij}^2(s, X(s), X(s - \tau)) ds, \end{aligned}$$

where $n \in \mathbb{N}$ is the dimension of the Brownian motion W . Let the function

$$(46) \quad \hat{b}_i(t, x, y) := b_i(t, x, y) - \int_{\mathcal{E}} g_i(t, x, y, \rho) \nu(d\rho), \quad \text{on } (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^d \times \mathbb{R}_+^d,$$

with $i = 1, 2, \dots, d$. Then, we have the following relationship between the local time L^i and the i th-regulator K^i .

Proposition 4.1. *For $i = 1, 2, \dots, d$, it holds that*

$$(47) \quad \frac{1}{2} L^i(t) = \int_0^t \mathbb{1}_{\{X^i(s)=0\}} \hat{b}_i(s, X(s), X(s - \tau)) ds + K^{i,c}(t), \quad \forall t \geq 0,$$

where $K^{i,c}(t)$ is the continuous counterpart of the i th-regulator $K^i(t)$. Moreover, if there exists a positive Borel measurable function $f_i(t, x)$ on $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^d$ and a positive Borel measurable function $l_i(x_i)$ on $x \in \mathbb{R}_+$ such that

$$(48) \quad \sum_{j=1}^n \sigma_{ij}^2(t, x, y) \geq l_i(x_i), \quad \left| \hat{b}_i(t, x, y) \right| \leq f_i(t, x),$$

with $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, then it holds that

$$(49) \quad \frac{1}{2}L^i(t) = K^{i,c}(t), \quad \forall t \geq 0.$$

Proof. By (1), the i th-element of the solution process X is given by

$$\begin{aligned} X^i(t) &= \xi^i(0) + \int_0^t b_i(s, X(s), X(s-\tau))ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(s, X(s), X(s-\tau))dW_j(s) \\ &\quad + \int_0^t \int_{\mathcal{E}} g_i(s, X(s-), X((s-\tau)-), \rho) \tilde{N}(d\rho, ds) + K^i(t) \\ &\geq 0, \quad \text{on } t \geq 0, \\ X^i(t) &= \xi^i(t), \quad \text{on } -\tau \leq t \leq 0. \end{aligned}$$

Obviously the process $(X^i(t); t \geq 0)$ is a r.c.l.l. $(\mathcal{F}_t; t \geq 0)$ -semimartingale. Using Tanaka's formula (see Protter [P04]), for $t \geq 0$, we have

$$\begin{aligned} X^i(t) &= \xi^i(0) + \int_0^t \mathbb{1}_{\{X^i(s-)>0\}}dX^i(s) + \sum_{0<s\leq t} \mathbb{1}_{\{X^i(s-)=0\}}X^i(s) + \frac{1}{2}L^i(t) \\ &= X^i(t) - \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}dX^i(s) + \sum_{0<s\leq t} \mathbb{1}_{\{X^i(s-)=0\}}X^i(s) + \frac{1}{2}L^i(t). \end{aligned}$$

As a consequence

$$\begin{aligned} \frac{1}{2}L^i(t) &= \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}dX^i(s) - \sum_{0<s\leq t} \mathbb{1}_{\{X^i(s-)=0\}}X^i(s) \\ &= \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}dX^i(s) - \sum_{0<s\leq t} \mathbb{1}_{\{X^i(s-)=0\}}[X^i(s) - X^i(s-)] \\ &= \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}dX^i(s) - \sum_{0<s\leq t} \mathbb{1}_{\{X^i(s-)=0\}}\Delta X^i(s) \\ &= \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}dX^{i,c}(s), \end{aligned}$$

where $X^{i,c}(t)$ corresponds to the continuous part of $X^i(t)$. Define the process with $i = 1, 2, \dots, d$,

$$M_i^W(t) = \sum_{j=1}^n \int_0^t \mathbb{1}_{\{X^i(s-)=0\}}\sigma_{ij}(s, X(s), X(s-\tau))dW_j(s), \quad t \geq 0.$$

Then, for $t \geq 0$,

$$(50) \quad \frac{1}{2}L^i(t) = \int_0^t \mathbb{1}_{\{X^i(s-) = 0\}} \hat{b}_i(s, X(s), X(s - \tau)) ds + M_i^W(t) + K^{i,c}(t),$$

which implies that the process $M_i^W = (M_i^W(t); t \geq 0)$ is of finite variation, since $K^{i,c}(t)$ is non-decreasing with respect to time $t \geq 0$. Note that M_i^W is also an $(\mathcal{F}_t; t \geq 0)$ -local martingale. Then it must hold that $M_i^W(t) = M_i^W(0) = 0$ for all times $t \geq 0$. By using (50) again, it follows that

$$\frac{1}{2}L^i(t) = \int_0^t \mathbb{1}_{\{X^i(s-) = 0\}} \hat{b}_i(s, X(s), X(s - \tau)) ds + K^{i,c}(t),$$

where the function $\hat{b}_i(\cdots)$ is defined as (46) with $i = 1, 2, \dots, d$. This shows the validity of (47).

Next we verify the validity of (49) under the condition (48). As a matter of fact, as a simple consequence of the occupation time formula (see Exercise (1.15) in Revuz and Yor [RM91]), we have, for $t \geq 0$,

$$\begin{aligned} & \int_0^t \mathbb{1}_{\{X^i(s) = 0\}} \left| \hat{b}_i(s, X(s), X(s - \tau)) \right| d \langle X^{i,c}, X^{i,c} \rangle_s \\ & \leq \int_0^t \mathbb{1}_{\{X^i(s) = 0\}} f_i(s, X(s)) d \langle X^{i,c}, X^{i,c} \rangle_s = \int_0^\infty \left(\int_0^t \mathbb{1}_{\{a=0\}} f_i(s, a) dL^{i,a}(s) \right) da \\ & = \int_0^\infty \left(\int_0^t \mathbb{1}_{\{a=0\}} f_i(s, 0) dL^i(s) \right) da = \left(\int_0^t f_i(s, 0) dL^i(s) \right) \int_0^\infty \mathbb{1}_{\{a=0\}} da \\ & = 0, \end{aligned}$$

where the nonnegative process $L^{i,a} = (L^{i,a}(t); t \geq 0)$ denotes the local time of the i th-element X^i of the solution process X at point $a \geq 0$. So that

$$\int_0^t \mathbb{1}_{\{X^i(s) = 0\}} \left| \hat{b}_i(s, X(s), X(s - \tau)) \right| l_i(X^i(s)) ds = 0, \quad \forall t \geq 0.$$

This yields that

$$\int_0^t \mathbb{1}_{\{X^i(s) = 0\}} \hat{b}_i(s, X(s), X(s - \tau)) ds = 0, \quad \forall t \geq 0,$$

which proves the validity of (49).

Remark 4.1. Here is an illustrative example for the condition (48) in the case of the dimension number $d = n = 1$. Let $(t, x, y, \rho) \in \mathbb{R}_+^3 \times \mathcal{E}$. We take the drift coefficient $b(t, x, y) = -\gamma(t)x + \theta_1(t)y$, the diffusion coefficient $\sigma(t, x, y) = l_1(x) + l_2(t, y)$ and the jump coefficient $g(t, x, y, \rho) = (\ell_g(t)x + \theta_2(t)y)h(t, \rho)$, where $\gamma(t), l_1(x), l_2(t, y), h(t, \rho) > 0$ and $\theta_1(t), \theta_2(t), \ell_g(t) \in \mathbb{R}$. For all $t \geq 0$, assume that $\ell_h(t) := \int_{\mathcal{E}} h(t, \rho) \nu(d\rho)$ is finite and the positive functions $x \rightarrow l_1(x)$ and $y \rightarrow l(t, y)$ are Lip-continuous with respect Lip-constants $\ell_1, \ell_2 > 0$. We take $\theta_1(t) = \theta_2(t)\ell_h(t)$ and choose appropriate set of parameters $(\gamma(t), \theta_2(t), \ell_g(t), \ell_h(t), \ell_1, \ell_2)$ such that (b, σ, g) satisfies the condition [A] (see the illustrative example presented in Section 1). In this case, we also have $|\hat{b}(t, x, y)| = |b(t, x, y) - \int_{\mathcal{E}} g(t, x, y, \rho) \nu(d\rho)| \leq |\gamma(t) + \ell_h(t) + \ell_g(t)|x$ and $\sigma^2(t, x, y) \geq \ell_1^2(x)$ with $(t, x, y) \in \mathbb{R}_+^3$. Thus, the condition (48) holds.

Remark 4.2. (1) Using the equalities (47) and (4), we can characterize the i th-regulator K^i in RSDDEJ (1) by the following way:

$$(51) \quad \begin{aligned} K^i(t) &= \frac{1}{2}L^i(t) - \int_0^t \mathbb{1}_{\{X^i(s)=0\}} \hat{b}_i(s, X(s), X(s-\tau))ds \\ &+ \sum_{0 < s \leq t} \left[\int_{\{s\}} \int_{\mathcal{E}} g_i(v, X(v-), X((v-\tau)-), \rho) N(d\rho, dv) + X^i(s-) \right]^- \end{aligned}$$

for all $t > 0$. If the condition (48) holds, then

$$(52) \quad K^i(t) = \frac{1}{2}L^i(t) + \sum_{0 < s \leq t} \left[\int_{\{s\}} \int_{\mathcal{E}} g_i(v, X(v-), X((v-\tau)-), \rho) N(d\rho, dv) + X^i(s-) \right]^-.$$

(2) If the jump coefficient g is nonnegative, then the regulator $K = (K^i(t); t \geq 0)_{d \times 1}$ has a continuous path modification by (4). In this case, assume the condition (48) is satisfied (see Remark 4.1), then by Proposition 4.1, it holds that

$$\begin{aligned} L^i(t) &= 2X^i(t) - 2\xi^i(0) - 2 \int_0^t b_i(s, X(s), X(s-\tau))ds - 2 \sum_{j=1}^n \int_0^t \sigma_{ij}(s, X(s), X(s-\tau), \rho) dW_j(s) \\ &+ 2 \int_0^t \int_{\mathcal{E}} g_i(s, X(s), X(s-\tau)) \tilde{N}(d\rho, ds). \end{aligned}$$

Hence, in the stationary setting, we have

$$\frac{\mathbb{E}[L^i(t)]}{t} = -\frac{2}{t} \int_0^t \mathbb{E}[b_i(s, \xi(0), \xi(-\tau))] ds, \quad t > 0.$$

If the drift function b is independent of time t , then

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{E}[L^i(t)]}{t} = -2\mathbb{E}[b_i(\xi(0), \xi(-\tau))].$$

The above quantity is usually called the loss rate in the reflected dynamics (see e.g., Asmussen [A03]).

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